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QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34. Given the mixed integral and functional equation
[*1917*, 134, 341; 1920, 114]

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. REPLY BY ELIJAH SWIFT, University of Vermont.

Assume that $f(x)$ is developable in a power series in the neighborhood of the origin

$$f(x) = A_0 + A_1x + \cdots + A_nx^n + \cdots.$$

Substituting this value for $f(x)$ and integrating we have

$$A_0h + A_1\frac{h^2}{2} + \cdots + A_n\frac{h^{n+1}}{n+1} + \cdots = A_0h + A_1\frac{h^2}{2} + \cdots + \frac{h}{6}A_n\left[4\cdot\frac{h^n}{2^n} + h^n\right] + \cdots;$$

or transposing,

$$\sum_{n=2}^{\infty} A_nh^{n+1}\left\{\frac{1}{n+1} - \frac{1}{6}\cdot\frac{4}{2^n} - \frac{1}{6}\right\} = 0.$$

If this is to hold for all values of h near 0, the coefficients of the terms of the series must all vanish. Obviously this is the case for $n = 2$ and $n = 3$. This leads to the equation

$$A_n\left(\frac{1}{n+1} - \frac{1}{6\cdot 2^{n-2}} - \frac{1}{6}\right) = 0.$$

Since

$$\frac{1}{n+1} - \frac{1}{6\cdot 2^{n-2}} - \frac{1}{6} < \frac{1}{n+1} - \frac{1}{6},$$

it is clear that for $n \geqq 5$, the coefficient of h^{n+1} cannot vanish unless $A_n = 0$. For $n = 4$, direct calculation gives the same result. Consequently *if the function $f(x)$ is analytic, it must be a polynomial of degree not larger than* 3.

II. REPLY BY A. A. BENNETT, Washington, D. C.

The problem as stated is ambiguous. The question depends upon whether h is to be some fixed value, as might be implied by the notation, or is a variable for all values of which the functional relation is to hold.

If h is constant, the solution obviously admits of an infinite number of independent parameters since only the definite integral and the value of the function at three points are involved. It is not difficult to select even an infinite number of values for which the above relation holds, while the function still retains an infinite number of independent parameters. In view of the application suggested, the following formulation of the problem may be of interest: To determine the character of $f(x)$ subject to the condition that the above relation holds for all values of h between 0 and H, (> 0), inclusive, and that $f(x)$ shall be defined in this interval as made up of a finite number of continuously joining analytic pieces—these pieces to be analytic even at the end points of the intervals in which they are used.

If h lies in the first interval (starting toward H from 0), it is at once verified that $f(x)$ is a polynomial of degree not greater than 3, but not otherwise restricted, so far as this interval is concerned. Expansion in series and comparison of undetermined coefficients suffices for this proof. Let a be the end point of this first interval. Let h be in the second interval, near a, so that $h/2$

is in the first interval. Suppose f_1, is the analytic continuation of $f(x)$ (defined in the first interval), which of course exists for the polynomial, and let f be the assumed function in the second interval. Now $f_1(x)$ will satisfy the condition of the problem as truly as $f(x)$. Subtracting, and denoting $f(x) - f_1(x)$ by $g(x)$, we have

$$\int_a^h g(x) = \frac{h}{6}[g(h)].$$

Expanding $g(h)$ in powers of $h - a$ and comparing coefficients to satisfy this condition, it is noted that $g(h)$ vanishes identically. Thus $f(x)$ coincides with $f_1(x)$ at the beginning of the second interval. In other words there is no break in the analytical representation. Thus a polynomial of at most the third degree but not otherwise specified is a solution and the only solution of the problem as restated.

III. Remarks by the Editor.

It is clear, as Professor Bennett states, that in case h is supposed to be a given constant, so little restriction is placed on the function $f(x)$ as to make the problem relatively uninteresting. When h is allowed to vary, however, it is still possible to interpret the problem in different ways, depending on the hypotheses to be made concerning $f(x)$. We may demand that $f(x)$ be analytic throughout a neighborhood of $x = 0$. This case has been considered thoroughly in both the preceding replies; all the algebraic steps are given in Professor Swift's reply. It follows, of course, that $f(x)$ must be a polynomial of degree $\leqq 3$.

We may demand that $f(x)$ consist at worst of a number of continuously joining analytic pieces. Professor Bennett shows that this extension produces no new solutions.

By his method it is possible to demonstrate also the following similar theorem: *If two functions which are continuous,* $0 \leqq x \leqq H$, *and satisfy the equation,* $0 \leqq h \leqq H$, *are identical in the interval* $0 \leqq x \leqq \epsilon$, *they are identical throughout the interval* $0 \leqq x \leqq H$.

To prove this, denote the difference of the two functions by $g(x)$, and let a be the greatest lower bound of the points for which $g(x) \neq 0$. Then for h slightly greater than a, we have, as in Professor Bennett's reply,

$$\int_a^h g(x)dx = \frac{h}{6}g(h).$$

From this equation it follows that $g(x)$ possesses a derivative, and that

$$hg'(h) - 5g(h) = 0;$$

hence

$$g(h) = Ah^5;$$

but as $g(a) = 0$, it follows that $g(h) = 0$. This contradicts the choice of the point a; no such point can exist and the theorem is proved.

In particular therefore, *if $f(x)$ is continuous,* $0 \leqq x \leqq H$, *and analytic at* $x = 0$, *it must be a polynomial of degree not greater than 3.*

It seems not unlikely that a similar result holds without the assumption of analyticity at any point. Two analogies make this suggestion worthy of consideration. In the first place, the equation which bears to the trapezoidal rule the same relation as the given equation to Simpson's rule:

$$\int_0^h f(x)dx = \frac{h}{2}[f(0) + f(h)]$$

is readily shown to have no solutions but linear functions, on the assumption of mere continuity. In the second place, if in the given equation both limits of integration, instead of the upper limit alone, be allowed to vary, the question may be answered without the use of analyticity. Let the equation

$$\int_\alpha^\beta f(x)dx = \frac{\beta - \alpha}{6}\left[f(\alpha) + 4f\left(\frac{\alpha + \beta}{2}\right) + f(\beta)\right]$$

be satisfied for all values of α and β in an interval throughout which $f(x)$ possesses a fifth derivative. By repeated differentiation we readily obtain

$$f^{\text{IV}}(\beta) = \frac{\beta - \alpha}{6}\left[\frac{1}{8}f^{\text{V}}\left(\frac{\alpha + \beta}{2}\right) + f^{\text{V}}(\beta)\right] + \frac{1}{6}\left[\frac{5}{4}f^{\text{IV}}\left(\frac{\alpha + \beta}{2}\right) + 5f^{\text{IV}}(\beta)\right].$$

Letting $\beta = \alpha$, we have $f^{\text{IV}}(\alpha) = 0$. Since this must hold for an arbitrary α in the interval under consideration, $f(x)$ can be at worst a cubic polynomial.

In view of the previous remarks, it seems reasonable to adopt as a goal with reference to this question the proof that if the equation holds for some range of the variable h, the function $f(x)$ can be only a polynomial of degree $\leqq 3$, under restrictions as light as possible—*e.g.*, that $f(x)$ should be continuous and possess a stated number (as small as it can be made) of derivatives.

DISCUSSIONS.

In the study of a geometric locus, it is frequently useful to have an analytic form of representation which shall exhibit the properties of the locus as regards its size and shape, without reference to its position. Such a form of representation, in the case of a plane curve, is furnished by the *intrinsic equation* of the curve —the functional relation which subsists between the length of arc measured from a fixed point of the curve to a variable point, and the radius of curvature at the variable point. Professor Light makes use of the intrinsic equation to study curves whose evolutes are similar to themselves. He obtains only the cases already known—the logarithmic spiral, and the cycloidal curves, and shows that no other examples exist whose intrinsic equations are of the special type $AR^n + BS^m - C = 0$. It would be of interest to know whether any other curves whatever can have this property.

In this department for May appeared a derivation of the formulæ for the tangents of the half angles of a triangle from the law of sines, by Professor Baudin, In the present number Professor Bohannan gives a method of obtaining tangent, sine and cosine of the half-angles by a geometric proof based on the use of inscribed and escribed circles.

Professor Poor recommends the more extensive use of directed lines and the method of projection in connection with elementary analytic geometry. In fact the use of such notions in the class-room is perhaps more prevalent than their appearance in text-books might indicate. Even in text-books however they are to be found here and there. The theorems stated by Professor Poor are now generally used in trigonometry to prove the formulæ for sines and cosines of sums and differences of angles. In some texts on analytic geometry they form the only mode of approach to transformation of coördinates by rotation; and they are frequently used in just the fashion suggested by Professor Poor to obtain the normal form of the equation of a straight line.

I. NOTE ON CURVES WHOSE EVOLUTES ARE SIMILAR CURVES.

BY G. H. LIGHT, University of Colorado.

It is known[1] that the evolutes of certain curves are similar curves. It is the purpose of this paper to give the general conditions that must be satisfied by

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[1] Euler: "Investigatio curvarum quae evolutae sui similes producunt," *Comm. Petrop.* vol. 12, 1750, pp. 19–23.

Loria: *Spezielle algebraische und transcendente ebene Kurven,* Leipzig, 1902, p. 625.

Cesàro, *Vorlesungen über naturliche Geometrie,* Leipzig, 1901, pp. 36–37.